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Consensus of second-order discrete-time multi-agent systems with fixed topology

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ABSTRACT

This paper studies the consensus of second-order discrete-time multi-agent systems with fixed topology. First, we formulate the problem and give some preliminaries. Then, by algebraic graph theory and matrix theory, the convergence of system matrix is analyzed. Our main results indicate that the consensus of second-order system can be achieved if and only if the topology graph has a directed spanning tree and the values of the scaling parameters satisfy a range. The eigenvalues of the corresponding Laplacian matrix play a key role in reaching consensus. Finally, numerical simulations are given to illustrate the results.

1. Introduction

In recent years, the multi-agent collective behaviors have attracted increasing attention due to growing interest in animal group behaviors and numerous issues have been addressed such as consensus problem, formation control, and so on. The consensus problem has attracted extensive attention due to its broad applications in cooperative control of unmanned air vehicles, formation control of mobile robots, control of communication networks, design of sensor networks, flocking of social insects, swarm-based computing and so on.

Convergence to a common value is called the consensus or agreement problem in the literatures, which has been investigated for a long time in different research fields, for example, in computer science [1], in management science and statistics [2], in system and control [3–5]. Consensus problem also has been studied in the context of cooperative control of multi-agent systems [7–9]. At present, the approaches of consensus analysis can be approximately summarized as two kinds. One is the algebraic method, that is, using algebraic graph theory and matrix theory to analyze the convergence of the product of system matrices [14,15]. Another approach is the system transformation method [27], that is, by defining new variables, the consensus problem of multi-agent systems can be equivalently converted into the stability problem of a corresponding reduced system, thus we can use the existing stability theory to solve the consensus of multi-agent systems. This simplifies the study of consensus problem.

In the past decade, numerous studies have been conducted on the consensus problem for agents with first-order dynamics [8–16]. A systematical framework of consensus problem in networked dynamic agents was established in [10] by Olfati-Saber and Murray. In [8,9], Jadbabaie et al. discussed the linearized Vicsek's model [6] and obtained that consensus can always be reached as long as the switching topology is periodically jointly connected or ultimately connected. In [15], Ren et al. extended the results of [8,10] and presented some more relaxable conditions for consensus of information under dynamically changing interaction topologies. [16] studied the average-consensus control problem.

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Recently, more attention has been paid to the consensus problem for agents with second-order dynamics [17–27]. In reality, a broad class of agents require a second-order dynamic model. For example, some agent dynamics such as holonomic mobile robot dynamic models can be feedback linearized as double integrators. Moreover, in contrast to the first-order consensus problem, it has been shown that consensus may fail to be achieved for agents with second-order dynamics even if the network topology has a directed spanning tree [27]. In [22,21], Ren et al. proposed a second-order protocol and provided sufficient conditions for systems with fixed and switching topologies. For a general consensus protocol, [23] established necessary and sufficient conditions to solve the consensus problem. [24] pointed out that the scaling parameters must satisfy some conditions for ensuring the consensus of second-order continuous-time systems with directed fixed topology, which depended on the real parts of the eigenvalues of the Laplacian matrix. In [25], Yu et al. proved that the consensus of second-order continuous-time systems with fixed topology can be reached if and only if the topology has a directed spanning tree and the values of the scaling parameters can satisfy a range. [25] also showed that not only the real parts but also the imaginary parts of the eigenvalues of the Laplacian matrix play key roles in reaching consensus. [27] proved that for discrete-time multi-agent systems with fixed and stochastic switching topology, there always exist scaling parameters such that the consensus conditions can be satisfied if and only if the graph (of the fixed topology) or the union of graphs (of the switching topology) has a directed spanning tree. Although [25] gave an explicit relation and a range of the scaling parameters under fixed topology, it only considered the continuous-time systems. [27] studied the second-order discrete-time systems, but it just proved the existence of the scaling parameters and did not solve the problem of how to choose the scaling parameters. This motivates us to write this paper.

Summarizing the above discussions, this paper will further investigate the consensus problem of second-order discrete-time multi-agent systems. First, we formulate the problem and give some preliminaries. Then, we analyze the convergence of system matrix, and derive the main results. Finally, an example is given to illustrate the results. Our results complement those results in [25,27] and can be regarded as an extension from continuous-time systems in [25] to discrete-time systems. But the extension is not trivial. To prove our results, the consensus problem of the discrete-time system is first equivalently converted into the Schur stability problem of a polynomial, by using a bilinear transformation, which can be further equivalently converted into the Hurwitz stability problem of another polynomial. Furthermore, using Hermite–Biehler Theorem [28], the consensus problem can finally be solved.

The remaining of this paper is organized as follows. In Section 2, some preliminaries on graph theory and model formulation are given. The consensus of second-order discrete-time multi-agent systems is discussed in Section 3. In Section 4, numerical examples are simulated to illustrate our results. Section 5 concludes the whole paper.

Notations. We use the following notations throughout this paper. Let I_N and O_N be N-dimensional identity and zero matrix, $1_N \in R^N$ and $0_N \in R^N$ be the vector with all entries being 1 and 0, $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ be the real and imaginary parts of a complex number u, respectively. $\otimes$ denotes Kronecker product of matrices.

2. Problem formulations and preliminaries

In this section, some basic knowledge on graph theory, problem formulations, some definitions and lemmas are given as the preliminaries of this paper.

2.1. Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order N, with the set of nodes $\mathcal{V} = (v_1, v_2, \ldots, v_N)$, and the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{N\times N}$ with nonnegative adjacency elements a_{ij}. A directed edge $\mathcal{E}_{ij}$ in $\mathcal{G}$ is denoted by the ordered pair of node (v_i, v_j), where v_i is defined as the parent node and v_j is defined as the child node, which means that node v_j can receive information from node v_i. The adjacency elements associated with the edges are positive, that is, $\mathcal{E}_{ij} \in \mathcal{E} \Longleftrightarrow a_{ji} > 0$. A graph is said to be balanced if $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ for all $i \in \mathcal{V}$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{V}$.

Correspondingly, the Laplacian matrix $L = [l_{ij}] \in R^{N\times N}$ of the directed graph is defined as

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j, \\ \sum_{k=1,k\neq i}^N a_{ik} & i = j. \end{cases}$$

An important property of L is that all the row sums of L are zero and thus 1_N is an eigenvector of L associated with the zero eigenvalue.

A directed path from node v_i to v_j is a sequence of edges (v_i, v_{i_1}), (v_{i_1}, v_{i_2}), $\ldots$, (v_{i_l}, v_j) in the directed graph with distinct nodes v_{i_k}, $k = 1, 2, \ldots, l$. A directed graph $\mathcal{G}$ is strongly connected if between any pair of distinct nodes v_i and v_j in $\mathcal{G}$, there is a directed path from v_i to v_j, $i, j = 1, 2, \ldots, N$. A root r is a node having the property that for each node v different from r, there is a directed path from r to v. A directed tree is a directed graph, in which there is exactly one root and every node except for this root has exactly one parent node. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathcal{G}$.

For a graph, if $(v_i, v_j) \in \mathcal{E}$, then $(v_j, v_i) \in \mathcal{E}$, then it is an undirected graph. A strongly connected undirected graph is only said to be connected. For an undirected graph, the adjacency matrix $\mathcal{A}$ is symmetric and thus every undirected graph is balanced.

2.2. Problem formulations

Consider a directed network with N agents which update their states based on information exchange. The topology of information exchange among agents can be described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Each agent in the network is a discrete-time second-order integrator given by

$$\begin{cases} x_i(k+1) = x_i(k) + v_i(k), \\ v_i(k+1) = v_i(k) + u_i(k), \quad i \in \ell =: \{1, 2, \ldots, N\}, \end{cases} \tag{1}$$

where $x_i \in R^n$ and $v_i \in R^n$ are the position and velocity states of agent i at time k, respectively.

We apply the consensus protocol as follows:

$$u_i(k) = k_1 \sum_{j=1}^{N} a_{ij}\big(x_j(k) - x_i(k)\big) + k_2 \sum_{j=1}^{N} a_{ij}\big(v_j(k) - v_i(k)\big), \quad i \in \ell, \tag{2}$$

where $k_1 > 0$ and $k_2 > 0$ are scaling parameters to be designed.

Substituting (2) into (1), system (1) can be rewritten as follows:

$$\begin{cases} x_i(k+1) = x_i(k) + v_i(k), \\ v_i(k+1) = v_i(k) - k_1 \displaystyle\sum_{j=1}^{N} l_{ij} x_j(k) - k_2 \sum_{j=1}^{N} l_{ij} v_j(k), \quad i \in \ell, \end{cases} \tag{3}$$

where l_{ij} $(i, j \in \ell)$ are the elements of Laplacian matrix L.

Let $x(k) = (x_1^T(k), \ldots, x_N^T(k))^T$, $v(k) = (v_1^T(k), \ldots, v_N^T(k))^T$, and $y(k) = (x^T(k), v^T(k))^T$, then system (3) can be rewritten in a compact matrix form as

$$y(k+1) = (\Gamma \otimes I_n) y(k), \tag{4}$$

where $\Gamma = \begin{pmatrix} I_N & I_N \\ -k_1 L & I_N - k_2 L \end{pmatrix}$.

Let $\phi_i(k) = x_i(k) - x_1(k)$, $\varphi_i(k) = v_i(k) - v_1(k)$, and $x(k) = (\phi_2^T(k), \ldots, \phi_N^T(k))^T$, $v(k) = (\varphi_2^T(k), \ldots, \varphi_N^T(k))^T$, $z(k) = (x^T(k), v^T(k))^T$, $i = 2, \ldots, N$, then we can obtain a reduced system

$$z(k+1) = (F \otimes I_n) z(k), \tag{5}$$

where $F = \begin{pmatrix} I_{N-1} & I_{N-1} \\ -k_1 \widetilde{L} & I_{N-1} - k_2 \widetilde{L} \end{pmatrix}$. $\widetilde{L} = \begin{pmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \cdots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{pmatrix}$ is defined as the reduced Laplacian matrix. Therefore, system (4) achieves consensus if and only if the reduced system (5) is stable [27].

Remark 1. The transformation from consensus problem to stability problem can simplify the study of consensus problem, since we can use the rich stability theory to solve the consensus problem of multi-agent systems.

2.3. Preliminaries

Definition 1. System (4) is said to be achieved consensus if for any initial conditions,

$$\lim_{k \to +\infty} \big\| x_i(k) - x_j(k) \big\| = 0, \qquad \lim_{k \to +\infty} \big\| v_i(k) - v_j(k) \big\| = 0, \quad \forall i, j \in \ell.$$

Definition 2. Let A be a square matrix, λ be an arbitrary eigenvalue of A, if there exists a vector ξ $(\xi \neq 0)$, such that

$$\xi^T A = \lambda \xi^T,$$

then ξ is called the left eigenvector of A associated with eigenvalue λ.

Remark 2. For the Laplacian matrix of a balanced graph, the vector of 1_N is not only an eigenvector but also a left eigenvector of L associated with eigenvalue 0.

Lemma 1. *(See [10, Remark 4, p. 1523].) The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues are positive if and only if the undirected network is connected.*

Lemma 2. *(See [25, Lemma 1, p. 1090].) The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.*

Lemma 3. *L has a zero eigenvalue with algebraic multiplicity m if and only if Γ has a 1 eigenvalue with algebraic multiplicity $2m$.*

Proof. Let λ be an eigenvalue of matrix Γ, μ_i $(i \in \ell)$ be eigenvalues of the Laplacian matrix L. Then we can get that

$$\begin{aligned}\det(\lambda I_{2N} - \Gamma) &= \det\begin{pmatrix} (\lambda-1)I_N & -I_N \\ k_1 L & (\lambda-1)I_N + k_2 L\end{pmatrix} = \det\bigl((\lambda-1)^2 I_N + \bigl(k_1 + (\lambda-1)k_2\bigr)L\bigr) \\ &= \prod_{i=1}^{N}\bigl((\lambda-1)^2 + \bigl(k_1 + (\lambda-1)k_2\bigr)\mu_i\bigr) = \prod_{i=1}^{N}\bigl(\lambda^2 - (2 - k_2\mu_i)\lambda + (1 + k_1\mu_i - k_2\mu_i)\bigr) = 0.\end{aligned}$$

Hence,

$$\lambda_{i1} = \frac{-k_2\mu_i + \sqrt{k_2^2\mu_i^2 - 4k_1\mu_i}}{2} + 1, \qquad \lambda_{i2} = \frac{-k_2\mu_i - \sqrt{k_2^2\mu_i^2 - 4k_1\mu_i}}{2} + 1, \quad i \in \ell. \tag{6}$$

From (6), it is easy to see that L has a zero eigenvalue of algebraic multiplicity m if and only if Γ has a 1 eigenvalue of algebraic multiplicity $2m$. □

Lemma 4. *The eigenvalues of the reduced Laplacian matrix $\widetilde{L}$ consist of the rest eigenvalues of Laplacian matrix L except a zero eigenvalue. Γ has two more 1 eigenvalues than F, and the rest eigenvalues are the same.*

Proof. The first part of this lemma can be obtained from the proof of Lemma 1 in [27]. Now we prove the second part of this lemma.

By the proof of Lemma 3, we get that

$$\det(\lambda I_{2(N-1)} - F) = \prod_{i=1}^{N-1}\bigl((\lambda-1)^2 + \bigl(k_1 + (\lambda-1)k_2\bigr)\widetilde{\mu}_i\bigr) = 0, \tag{7}$$

$$\det(\lambda I_{2N} - \Gamma) = \prod_{i=1}^{N}\bigl((\lambda-1)^2 + \bigl(k_1 + (\lambda-1)k_2\bigr)\mu_i\bigr) = 0, \tag{8}$$

where $\widetilde{\mu}_i$ $(i = 1, \ldots, N-1)$, μ_i $(i = 1, \ldots, N)$ denote the eigenvalues of $\widetilde{L}$ and L, respectively.

Then, we can easily obtain that Γ has two more 1 eigenvalues than F, and the rest eigenvalues are the same. Hence, Lemma 4 holds. □

Lemma 5. *(See [29, Lemma 3, p. 3905].) The polynomial $\gamma(\sigma)$ is Hurwitz stable if and only if the related pair $m(\omega)$, $n(\omega)$ is interlaced, and $m(0)n'(0) - m'(0)n(0) > 0$, where $m(\omega)$, $n(\omega)$ are the real and imaginary parts of $\gamma(i\omega)$, respectively.*

3. Consensus of second-order discrete-time multi-agent systems

In this section, we study the consensus of system (4) and give the main results.

Theorem 1. *The consensus of multi-agent system (4) can be achieved if and only if the matrix Γ has exactly a 1 eigenvalue of multiplicity two and all the other eigenvalues are in the unit circle. In addition, if the consensus is reached, then $\|x_i(k) - \sum_{j=1}^{N}\xi_j x_j(0) - \sum_{j=1}^{N}\xi_j v_j(0)k\| \to 0$, $\|v_i(k) - \sum_{j=1}^{N}\xi_j v_j(0)\| \to 0$ as $k \to +\infty$, where ξ is the unique nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T 1_N = 1$.*

Proof. (Sufficiency.) Note that 1 is an eigenvalue of matrix Γ with multiplicity 2. From calculation of $\Gamma\varphi = \varphi$, where φ is an unit right eigenvector of matrix Γ associated with eigenvalue 1, then we can easily obtain that $\varphi = (1_N^T, 0_N^T)^T/\sqrt{N}$, which is unique. Because there is only one unique eigenvector of matrix Γ associated with eigenvalue 1, so the corresponding Jordan block cannot be diagonal. Then there exists a nonsingular matrix $P \in R^{2N\times 2N}$, such that $P^{-1}\Gamma P = J$, where $J =$

$\begin{pmatrix} 1 & 1 & 0_{1\times(2N-2)} \\ 0 & 1 & 0_{1\times(2N-2)} \\ 0_{(2N-2)\times 1} & 0_{(2N-2)\times 1} & \widetilde{J} \end{pmatrix}$ is the Jordan canonical form associated with Γ, and $\widetilde{J}$ is the upper diagonal Jordan block matrix. So we can obtain

$$\Gamma = PJP^{-1} = (\zeta_1, \ldots, \zeta_{2N}) \begin{pmatrix} 1 & 1 & 0_{1\times(2N-2)} \\ 0 & 1 & 0_{1\times(2N-2)} \\ 0_{(2N-2)\times 1} & 0_{(2N-2)\times 1} & \widetilde{J} \end{pmatrix} \begin{pmatrix} \eta_1^T \\ \vdots \\ \eta_{2N}^T \end{pmatrix},$$

where ζ_j and η_j $(j = 1, 2, \ldots, 2N)$ are the right and left eigenvectors or generalized eigenvectors of Γ, respectively.

Since $\zeta_1 = (1_N^T, 0_N^T)^T$ is the unique eigenvector of Γ associated with eigenvalue 1, then from

$$(\Gamma - I_{2N})\zeta_2 = \zeta_1,$$

that is,

$$\begin{pmatrix} O_N & I_N \\ -k_1 L & -k_2 L \end{pmatrix} \zeta_2 = \begin{pmatrix} 1_N \\ 0_N \end{pmatrix},$$

we can get $\zeta_2 = (0_N^T, 1_N^T)^T$. Therefore, ζ_2 is the generalized right eigenvector of matrix Γ associated with eigenvalue 1.

Therefore, one can easily obtain the generalized left eigenvector $\eta_1 = (\xi^T, 0_N^T)^T$ and the left eigenvector $\eta_2 = (0_N^T, \xi^T)^T$ of matrix Γ associated with eigenvalue 1, where ξ is defined as in the statement of Theorem 1.

From (4), we get

$$y(k+1) = (\Gamma \otimes I_n) y(k) = (\Gamma \otimes I_n)^{k+1} y(0).$$

Furthermore, by the properties of the Kronecker product, we have

$$\begin{aligned}
(\Gamma \otimes I_n)^k &= \left[\left(PJP^{-1}\right) \otimes I_n\right]^k = \left[\left(PJP^{-1}\right) \otimes \left(PI_nP^{-1}\right)\right]^k = \left[(P \otimes P)(J \otimes I_n)\left(P^{-1} \otimes P^{-1}\right)\right]^k \\
&= (P \otimes P)(J \otimes I_n)^k\left(P^{-1} \otimes P^{-1}\right) = (P \otimes P)\left(J^k \otimes I_n\right)\left(P^{-1} \otimes P^{-1}\right) = \left(PJ^kP^{-1}\right) \otimes I_n \\
&= \left\{ P \begin{pmatrix} 1 & k & 0_{1\times(2N-2)} \\ 0 & 1 & 0_{1\times(2N-2)} \\ 0_{(2N-2)\times 1} & 0_{(2N-2)\times 1} & \widetilde{J}^k \end{pmatrix} P^{-1} \right\} \otimes I_n.
\end{aligned}$$

It is easy to get that

$$\lim_{k\to+\infty} \widetilde{J}^k = 0_{(2N-2)\times(2N-2)}.$$

Hence, we have

$$\begin{aligned}
&\lim_{k\to+\infty} \left\| \begin{pmatrix} x(k) \\ v(k) \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \otimes 1_N \right\| \\
&= \lim_{k\to+\infty} \left\| (\Gamma \otimes I_n)^k \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \otimes 1_N \right\| \\
&= \lim_{k\to+\infty} \left\| \left[\begin{pmatrix} 1_N\xi^T & k1_N\xi^T \\ O_N & 1_N\xi^T \end{pmatrix} \otimes I_n \right] \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \otimes 1_N \right\| \\
&= \lim_{k\to+\infty} \left\| \begin{pmatrix} \xi_1 I_n & \cdots & \xi_N I_n & k\xi_1 I_n & \cdots & k\xi_N I_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi_1 I_n & \cdots & \xi_N I_n & k\xi_1 I_n & \cdots & k\xi_N I_n \\ O_n & \cdots & O_n & \xi_1 I_n & \cdots & \xi_N I_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O_n & \cdots & O_n & \xi_1 I_n & \cdots & \xi_N I_n \end{pmatrix} \begin{pmatrix} x_1(0) \\ \vdots \\ x_N(0) \\ v_1(0) \\ \vdots \\ v_N(0) \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \vdots \\ \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \sum_{j=1}^N \xi_j v_j(0) \\ \vdots \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \right\| \\
&= \lim_{k\to+\infty} \left\| \begin{pmatrix} \xi_1 x_1(0) + \cdots + \xi_N x_N(0) + k\xi_1 v_1(0) + \cdots + k\xi_N v_N(0) \\ \vdots \\ \xi_1 x_1(0) + \cdots + \xi_N x_N(0) + k\xi_1 v_1(0) + \cdots + k\xi_N v_N(0) \\ \xi_1 v_1(0) + \cdots + \xi_N v_N(0) \\ \vdots \\ \xi_1 v_1(0) + \cdots + \xi_N v_N(0) \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \vdots \\ \sum_{j=1}^N \xi_j (x_j(0) + v_j(0)k) \\ \sum_{j=1}^N \xi_j v_j(0) \\ \vdots \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \right\| = 0,
\end{aligned}$$

which indicates that the consensus of system (4) is achieved.

(Necessity.) Now, we prove the necessity by contradiction. If the condition that matrix Γ has exactly a 1 eigenvalue of multiplicity two and all the other eigenvalues are in the unit circle is not satisfied, then by Lemma 3, the multiplicity of 1 eigenvalue in Γ is at least 2 since L has a zero eigenvalue at least. Hence, there are three cases needed to be discussed:

Case I: The multiplicity of 1 eigenvalue in Γ is 2, and there exists at least an eigenvalue which is not in the unit circle;

Case II: The multiplicity of 1 eigenvalue in Γ is more than 2, and the rest eigenvalues are in the unit circle;

Case III: The multiplicity of 1 eigenvalue in Γ is more than 2, and there exists at least an eigenvalue which is not in the unit circle.

For Case I, by Lemma 4, if Γ has an eigenvalue which is not in the unit circle, then F also has an eigenvalue which is not in the unit circle. Therefore, the stability of system (5) cannot be achieved, which means that the consensus of system (4) cannot be achieved. Similarly, we can prove Case II and Case III. Hence, this theorem holds. $\square$

Theorem 2. *The consensus of multi-agent system* (4) *can be achieved if and only if the topology graph contains a directed spanning tree and*

$$\begin{cases} k_1 - 2k_2 > \dfrac{-4\,\mathrm{Re}(\mu_i)}{|\mu_i|^2}, \\ k_2 > k_1 > 0, \\ [(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)](k_2 - k_1)^2 > \dfrac{4k_1\,\mathrm{Im}^2(\mu_i)}{|\mu_i|^2}, \end{cases} \tag{9}$$

where μ_i are the nonzero eigenvalues of the Laplacian matrix L, $i = 2, 3, \dots, N$. In addition, if the consensus of system (4) *is reached, then $\|x_i(k) - \sum_{j=1}^{N} \xi_j x_j(0) - \sum_{j=1}^{N} \xi_j v_j(0)k\| \to 0$, $\|v_i(k) - \sum_{j=1}^{N} \xi_j v_j(0)\| \to 0$ as $k \to +\infty$, where ξ is the unique nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T 1_N = 1$.*

Proof. (Necessity.) If the consensus of second-order system (4) can be achieved, then by Theorem 1, Γ has exactly a 1 eigenvalue of multiplicity two and all the other eigenvalues are in the unit circle. Moreover, by Lemma 3, L has a simple 0 eigenvalue. Let $\lambda - 1 = s$, where λ is an eigenvalue of Γ and $|\lambda| < 1$, then (8) can be rewritten as $s^2 + (k_1 + sk_2)\mu_i = 0$, that is, $s^2 + k_2\mu_i s + k_1\mu_i = 0$ $(i = 2, \dots, N)$. Hence, $\mathrm{Re}(s) < 0$. Furthermore, $\mathrm{Re}(s_1) < 0$ and $\mathrm{Re}(s_2) < 0$, where s_1, s_2 are the roots of $s^2 + k_2\mu_i s + k_1\mu_i = 0$. Since $s_1 + s_2 = -k_2\mu_i$ and $k_2 > 0$, then $\mathrm{Re}(s_1 + s_2) = \mathrm{Re}(-k_2\mu_i) < 0$. Therefore, $\mathrm{Re}(\mu_i) > 0$ $(i = 2, \dots, N)$. Then by Lemma 2, the topology graph contains a directed spanning tree.

Define $g(\lambda) = (\lambda - 1)^2 + (k_1 + (\lambda - 1)k_2)\mu_i$ $(i = 2, \dots, N)$. By (8) and Theorem 1, system (4) is consensus if and only if $g(\lambda)$ is Schur stable. Applying the bilinear transformation $\sigma = \varphi(\lambda) = \dfrac{\lambda + 1}{\lambda - 1}$ to $g(\lambda)$, we get a new polynomial

$$\begin{aligned} \theta(\sigma) &= (\sigma - 1)^2 g\left(\frac{\sigma + 1}{\sigma - 1}\right) = (\sigma - 1)^2 \left\{\left(\frac{\sigma + 1}{\sigma - 1} - 1\right)^2 + \left[k_1 + \left(\frac{\sigma + 1}{\sigma - 1} - 1\right)k_2\right]\mu_i\right\} \\ &= (\sigma - 1)^2 \left[\left(\frac{2}{\sigma - 1}\right)^2 + \left(k_1 + \frac{2}{\sigma - 1}k_2\right)\mu_i\right] = 4 + k_1\mu_i(\sigma - 1)^2 + 2k_2\mu_i(\sigma - 1) \\ &= k_1\mu_i\sigma^2 + 2\mu_i(k_2 - k_1)\sigma + (k_1\mu_i - 2k_2\mu_i + 4). \end{aligned}$$

Define $\gamma(\sigma)$ as

$$\gamma(\sigma) = \frac{\theta(\sigma)}{k_1\mu_i} = \sigma^2 + \frac{2(k_2 - k_1)}{k_1}\sigma + \frac{k_1\mu_i - 2k_2\mu_i + 4}{k_1\mu_i},$$

then the polynomial $g(\lambda)$ is Schur stable if and only if the polynomial $\gamma(\sigma)$ is Hurwitz stable [29].

Let $\sigma = i\omega$, then we get

$$\gamma(i\omega) = (i\omega)^2 + \frac{2(k_2 - k_1)}{k_1}(i\omega) + \frac{(k_1 - 2k_2)|\mu_i|^2 + 4\overline{\mu_i}}{k_1|\mu_i|^2}.$$

It follows that

$$m(\omega) = -\omega^2 + \frac{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}{k_1|\mu_i|^2}, \tag{10}$$

$$n(\omega) = \frac{2(k_2 - k_1)}{k_1}\omega - \frac{4\,\mathrm{Im}(\mu_i)}{k_1|\mu_i|^2}. \tag{11}$$

Then, by Lemma 5, $\gamma(\sigma)$ is Hurwitz stable if and only if the following conditions hold:

(a) the polynomial $m(\omega)$ has two distinct roots $m_1 < m_2$, that is,

$$\Delta = \frac{4[(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)]}{k_1|\mu_i|^2} > 0;$$

(b) the interlaced condition holds, that is, $m_1 < n_1 < m_2$, where n_1 is the unique root of the polynomial $n(\omega)$;
(c) $m(0)n'(0) - m'(0)n(0) > 0$.
From (10), we have

$$m_1 = -\frac{\sqrt{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}}{\sqrt{k_1}|\mu_i|}, \qquad m_2 = \frac{\sqrt{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}}{\sqrt{k_1}|\mu_i|}.$$

From (11), we have

$$n_1 = \frac{2\,\mathrm{Im}(\mu_i)}{(k_2 - k_1)|\mu_i|^2}.$$

From (c), we obtain

$$\frac{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}{k_1|\mu_i|^2} \cdot \frac{2(k_2 - k_1)}{k_1} > 0,$$

then $k_2 > k_1 > 0$, since $\Delta > 0$.
Therefore,

$$\begin{cases} \dfrac{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}{k_1|\mu_i|^2} > 0, \\ k_2 > k_1 > 0, \\ -\dfrac{\sqrt{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}}{\sqrt{k_1}|\mu_i|} < \dfrac{2\,\mathrm{Im}(\mu_i)}{(k_2 - k_1)|\mu_i|^2} < \dfrac{\sqrt{(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)}}{\sqrt{k_1}|\mu_i|}, \end{cases}$$

$$\iff \quad \begin{cases} k_1 - 2k_2 > \dfrac{-4\,\mathrm{Re}(\mu_i)}{|\mu_i|^2}, \\ k_2 > k_1 > 0, \\ \big[(k_1 - 2k_2)|\mu_i|^2 + 4\,\mathrm{Re}(\mu_i)\big](k_2 - k_1)^2 > \dfrac{4k_1\,\mathrm{Im}^2(\mu_i)}{|\mu_i|^2}. \end{cases}$$

(Sufficiency.) From the above proof in necessity, one obtains that if k_1, k_2 satisfy (9), then the roots of $g(\lambda) = 0$ are in the unit circle, which means that the eigenvalues of Γ are in the unit circle except 1. Because the topology contains a directed spanning tree, 0 is a simple eigenvalue of L. Therefore, by Lemma 3, Γ has a 1 eigenvalue of multiplicity two.

By Theorem 1, if the consensus of second-order system (4) can be achieved, then $\|x_i(k) - \sum_{j=1}^N \xi_j x_j(0) - \sum_{j=1}^N \xi_j v_j(0)k\| \to 0$, $\|v_i(k) - \sum_{j=1}^N \xi_j v_j(0)\| \to 0$ as $k \to +\infty$. Hence, Theorem 2 holds. □

Remark 3. From (9), it is found that both real and imaginary parts of the eigenvalues of Laplacian matrix play important roles in reaching consensus. And by Theorem 2 in [27], we can obtain that the formula of (9) must has a solution at least.

Remark 4. In the proof of Theorem 2, we only consider the scaling parameters without choosing connection weights. This implies that we can make the system achieve consensus by designing the scaling parameters under different connection weights. Of course, the choosing of the scaling parameters still relies on the values of the connection weights.

The undirected graph can be treated as a special directed graph. Therefore, by Lemma 1 and Theorem 2, we can easily get the following corollary.

Corollary 1. *If the topology is an undirected graph, then the consensus of system* (4) *can be achieved if and only if the topology graph is connected and*

$$\begin{cases} k_2 > k_1 > 0, \\ k_1 - 2k_2 > \dfrac{-4}{\mu_i}, \end{cases} \tag{12}$$

where μ_i are the nonzero eigenvalues of the Laplacian matrix L, $i = 2, 3, \ldots, N$. In addition, if the consensus of system (4) *is reached, then $\|x_i(k) - \frac{1}{\sqrt{N}}\sum_{j=1}^N x_j(0) - \frac{1}{\sqrt{N}}\sum_{j=1}^N v_j(0)k\| \to 0$, $\|v_i(k) - \frac{1}{\sqrt{N}}\sum_{j=1}^N v_j(0)\| \to 0$ as $k \to +\infty$.*

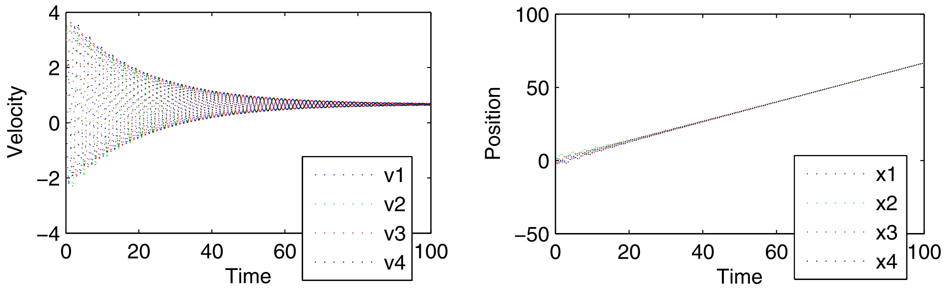

Fig. 1. Velocity and position of agents, where $k_1 = 0.1$, $k_2 = 1$.

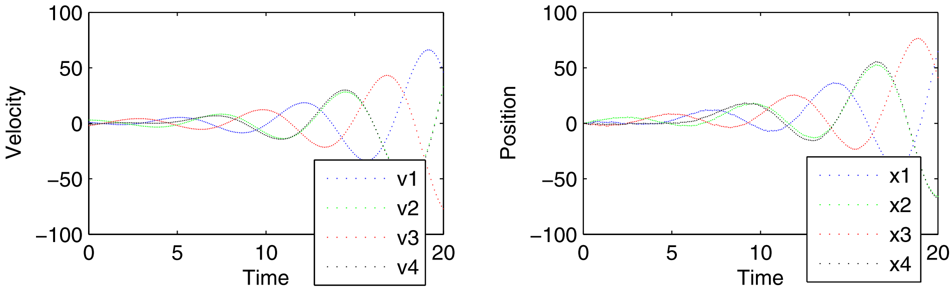

Fig. 2. Velocity and position of agents, where $k_1 = 0.4$, $k_2 = 0.5$.

4. Numerical simulations

In this section, we give an example to illustrate the theoretical results obtained in the previous sections. The topology graph in our simulations has 0–1 weight.

Consider the system (4) with 4 agents. The Laplacian matrix L is $\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ and its four eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1.5 + 0.866i$, $\lambda_4 = 1.5 - 0.866i$. By Lemma 2, the topology graph has a directed spanning tree.

Let $k_1 = 0.1$, $k_2 = 1$, then k_1, k_2 satisfy (9), and the consensus of system (4) can be achieved. The velocity and position states of all the agents are shown in Fig. 1.

If $k_1 = 0.4$, $k_2 = 0.5$, then k_1, k_2 do not satisfy (9). Hence, the consensus of system (4) cannot be achieved. The velocity and position states of all the agents are shown in Fig. 2.

5. Conclusions

The consensus of second-order systems depends not only on topology conditions but also on the scaling parameters. In this paper, detailed analysis has been performed on the case that the second-order discrete-time dynamics of each agent are determined by position and velocity terms. Through analyzing the eigenvalues of system matrix, a necessary and sufficient condition has been established to ensure the consensus of second-order systems. Then, an example is given to illustrate the obtained results. Future work includes more complicated and realistic agent dynamics. For example, it is of great interest to generalize the result of this paper to the time-varying topology case.

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